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THE CHROMATIC NUMBER OF THE PRODUCT OF TWO $\aleph_1$-CHROMATIC GRAPHS CAN BE COUNTABLE

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We prove (in ZFC) that for every infinite cardinal $\varkappa$ there are two graphs G_0, G_1 with $\chi(G_0)=\chi(G_1)=\varkappa^+$ and $\chi(G_0\times G_1)=\varkappa$. We also prove a result from the other direction. If $\chi(G_0)\geqq \aleph_0$ and $\chi(G_1)=k<\omega$, then $\chi(G_0\times G_1)=k$.

0. Introduction, Notation, Problems

A graph $G=\langle V(G), E(G)\rangle=\langle V, E\rangle$ is a structure of the form $E\subset[V]^2$. $\chi(G)$ denotes the chromatic number of G.

If $G_i=\langle V_i, E_i\rangle$ $(i<2)$ are graphs, $G_0\times G_1$ is the graph $\langle V_0\times V_1, E_0 * E_1\rangle$ where $V_0\times V_1$ is the usual Cartesian product, for $u\in V_0\times V_1, u_0\in V_0, u_1\in V_1, u=\langle u_0, u_1\rangle$ and for $u, v\in V_0\times V_1$ $\{u, v\}\in E_0 * E_1$ iff $\{u_i, v_i\}\in E_i$ for $i<2$.

It is clear from the definitions that

(1) $\chi(G_0\times G_1)\leqq\min\left(\chi(G_0), \chi(G_1)\right)$ holds for all graphs G_i $i<2$.

It was conjectured in Hedetniemi's paper [2] that at least for finite graphs equality holds in (1). This is obvious if $\chi(G_i)\leqq 3$ for $i<2$, and the conjecture is proved in a paper of this issue ([1]) in case $\chi(G_i)=4$ $(i<2)$ by M. El-Zahar and N. Sauer. The main aim of this note is to point out that for infinite graphs equality does not always hold in (1).

Theorem 1. *For every infinite cardinal $\varkappa$, there are graphs G_i, $i<2$ with $\chi(G_i)=\varkappa^+$ and such that*

$$\chi(G_0\times G_1) = \varkappa.$$

On the other hand it is quite easy to show that in (1) equality holds if $\chi(G_0)=\aleph_0$ and $\chi(G_1)<\aleph_0$ (see Theorem 2). (Here $\aleph_0$ can be replaced by a strongly compact cardinal.)

The question arises if $\varkappa^+$ can be replaced in Theorem 1 by some larger cardinal. Here the answer seems to depend on the set theory we are working with.

It was proved by L. Sokoup that the following is consistent with ZFC+GCH:

(2) *There are graphs G_i $i<2$ with $\chi(G_i)=|V_i|=\aleph_2$ such that $\chi(G_0\times G_1)=\aleph_0$.*

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Note that (2) easily implies that each subgraph of size $\aleph_1$ of say G_0 has chromatic number $\aleph_0$, hence by a result of Laver and Foreman (2) can not be proved in ZFC+GCH.

Problem. *Is it consistent with ZFC+GCH that there are graphs with* $\chi(G_0)=\chi(G_1)\geqq\aleph_\omega$ *and* $\chi(G_0\times G_1)<\aleph_\omega$?

1. Proofs

Let $\varkappa\geqq\omega$, $A\subset\varkappa^+$. Set

$$V_A(\varkappa)=V_A=\{f\colon f \text{ is a function } \wedge f \text{ is one-to-one} \wedge D(f)\in A\wedge R(f)\subset\varkappa\};$$

$$f\prec g\Leftrightarrow f\subsetneqq g.$$

$\langle V_A, \prec\rangle$ is a tree (in the set theoretic sense i.e. for each $f\in V_A$ the set $\{g\in V_A\colon g\preceq f\}$ is well-ordered). The following is an unpublished but well-known result of F. Galvin and R. Laver.

Theorem. *Assume A is a stationary subset of* $\varkappa^+$. *Then the tree* $\langle V_A, \prec\rangle$ *is not* $\varkappa$-*special i.e. it is not the union of* $\leqq\varkappa$ *antichains.* ∎

The comparability graph Comp $\langle V_A, \prec\rangle$ of the tree $\langle V_A, \prec\rangle$ is $G_A=\langle V_A, E_A\rangle$ where $E_A=\{\{f,g\}\in[V_A]^2\colon f\prec g\vee g\prec f\}$, and the fact that $\langle V_A, \prec\rangle$ is not $\varkappa$-special means exactly that $\chi(G_A)\geqq\varkappa^+$.

Proof of Theorem 1. Let A_0, A_1 be two disjoint stationary subsets of $\varkappa^+$. By the Galvin—Laver result it is sufficient to see that for $G=G_{A_0}\times G_{A_1}$, $\chi(G)\leqq\varkappa$ holds. Now $V_{A_0}\times V_{A_1}=K_0\cup K_1$ where

$$K_0=\{\langle f_0,f_1\rangle\in V_{A_0}\times V_{A_1}\colon D(f_0)<D(f_1)\}$$

$$K_1=\{\langle f_0,f_1\rangle\in V_{A_0}\times V_{A_1}\colon D(f_1)<D(f_0)\}.$$

By symmetry it is sufficient to see that G has a good coloring with $\varkappa$ colors on K_0. Define $F\colon K_0\to\varkappa$ by $F(f_0,f_1)=f_1(D(f_0))$. To see that F is a good coloring let $\langle f_0,f_1\rangle,\langle g_0,g_1\rangle\in K_0$, $\{\langle f_0,f_1\rangle,\langle g_0,g_1\rangle\}\in E_{A_0}*E_{A_1}$. Set $D(f_i)=\alpha_i$, $D(g_i)=\beta_i$ for $i<2$. Then $\alpha_0<\alpha_1,\beta_0<\beta_1$ and we may assume $\alpha_1<\beta_1$. $\{f_0,g_0\}\in E_{A_0}$ implies $\alpha_0\neq\beta_0$. $\{f_1,g_1\}\in E_{A_1}$ implies $f_1(\alpha_0)=g_1(\alpha_0)$. Hence $F(f_0,f_1)=f_1(\alpha_0)=g_1(\alpha_0)\neq g_1(\beta_0)=F(g_0,g_1)$ because g_1 is one-to-one. ∎

S. Todorčević pointed out to me the following facts.

A) Under some set theoretical hypothesis (say $\diamondsuit$) one can construct graphs $|G_0|=|G_1|=\aleph_1$ not containing independent sets of size $\aleph_1$ and such that $\chi(G_0\times G_1)\leqq\aleph_0$.

B) Assume S and T are trees. Let $S\otimes T$ denote $\{\langle s,t\rangle\in S\times T$ with $ht(s)=ht(t)\}$ with the Cartesian ordering. $S\otimes T$ is a tree, if $S\otimes T$ $\varkappa$-special then Comp $(S)\times$Comp (T) has chromatic number at most $\varkappa$, and this gives many new examples establishing Theorem 1. Some examples of such trees can be found in his paper [3].

Finally we prove

Theorem 2. *Assume* $\chi(G_0) \geqq \aleph_0$, $\chi(G_1) = k+1$, $k < \omega$. *Then* $\chi(G_0 \times G_1) = k+1$.

Proof. Let $G_0 = \langle V_0, E_0 \rangle$, $G_1 = \langle V_1, E_1 \rangle$. By compactness, we may assume $|V_1| < \aleph_0$.
Let $f\colon V_0 \times V_1 \to k$ be a good k-coloring of $G_0 \times G_1$. For $x \in V_0$ let $f_x(y) = f(x, y)$ for $y \in V_1$. By the assumption, f_x is not a good coloring of G_1, hence if $f_x = f_{x'}$ for $x \neq x' \in V_0$, then $\{x, x'\} \notin E_0$. This implies that

$$\chi(G_0) \leqq k^{|V_1|} < \aleph_0. \quad \blacksquare$$

References

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